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Explicit Form of the Time Operator of a Gaussian Stationary Process

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We present the time operator theory in the framework of stationary stochastic processes. The main results of the paper is the derivation of the time operator acting on the Fock space associated with a discrete time gaussian stationary process.

KEY WORDS: time operator; gaussian stationary process.

1. INTRODUCTION

Let $\{V_t\}_{t\in I}, \subset \mathbb{R}$, be a semigroup of isometries on a Hilbert space $\mathcal{H}$. if T is a selfadjoint operator on $\mathcal{H}$ such that each V_t preserves the domain of T and

$$T V_t = V_t T + t V_t \tag{1}$$

on a dense subspace of $\mathcal{H}$, then T is called the *time operator* of $\{V_t\}$.

The semigroup $\{V_t\}$ may describe the time evolution of some physical system, in particular, a dynamical system. If such time evolution is reversible, then $\{V_t\}$ is a group of unitary operators.

Time operators in dynamical systems were introduced by Misra (1978) and Prigogine (1980) for the study of irreversible behavior of highly unstable reversible dynamics. It turns out, however, that time operators can also used as a new tool for the spectral analysis of various evolution semigroups.

A crucial step in the spectral analysis of an evolution semigroup $\{V_t\}$ on a Hilbert space $\mathcal{H}$ with the use of its time operator T is to find the spectral family $\{E_t\}$ of T which is, according to (1), related with $\{V_t\}$ by the relation

$$E_{s+t} V_t = V_t E_s$$

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In the case of discrete time, $I \subset \mathbb{Z}$, the task is to find the representation

$$T = \sum_n n \sum_k |\varphi_{n,k}\rangle\langle\varphi_{n,k}|$$

where $\{\varphi_{n,k}\}$ is a complete family of eigenvectors of T, $T\varphi_{n,k} = n\varphi_{n,k}$, and V_t is the shift $V_t\varphi_{n,k} = \varphi_{n+t,k}$. In the latter case the action of V_t on an element $h \in \mathcal{H}$, $h = \sum_{n,k} a_{n,k}\varphi_{n,k}$, is nothing but a shift of its representation, $V_t h = \sum_{n,k} a_{n,k}\varphi_{n+t,k} = \sum_{n,k} a_{n-t,k}\varphi_{n,k}$. The knowledge of the eigenvectors of T allows therefore to solve the prediction problem for the evolution semigroup $\{V_t\}$. The spaces $\mathcal{N}_n$ spanned by the eigenvectors $\varphi_{n,\alpha}$ are called the age eigenspaces or the spaces of innovations at time n, as they correspond to the new information, or detail, brought at time n (see Suchanecki and Antoniou, 2003, for more details).

The time operator method of spectral analysis can be obviously applied only for those evolution semigroups for which time operators exist and can be explicitly constructed. A serious obstacle is that the class of semigroups that admit time opeators is relatively narrow. Even if a time operator exists its explicit construction is in general a nontrivial task.

Time operators have been initially constructed for some evolution semigroups arising from dynamical systems. Namely, for those which have the strongest ergodic properties: Kolmogorov and exact systems (Misra *et al.*, 1979; Suchanecki and Weron, 1990; Suchanecki, 1992; Antoniou and Suchanecki, 2000; Antoniou *et al.*, 1999). Then time operators have been associated with other semigroups like the diffusion semigroup (Antoniou *et al.*, 2000), with approximations (Suchanecki and Antoniou, 2003), with wavelets (Antoniou and Gustafson, 2000; Antoniou and Suchanecki, 2000), and with Markov semigroups (Antoniou and Suchanecki, 2003a). Some connections of time operators with wide sense stationary processes have also been discussed (Antoniou and Gustafson, 1999).

In this article, we shall focus our attention on semigroups that arise from stationary stochastic processes. the main reason of considering stationary processes is that they can be described in terms of groups of shift operators. This property is necessary to link time operators with stochastic processes, although not sufficient. An additional assumption that the process is innovative (or an innovation stochastic process) must be added (Antoniou and Suchanecki, 2003b). An example of a discrete time innovation process is a sequence of independent, identically distributed random variables. Innovative are also completely nondeterministic stationary processes, in particular, gaussian stationary processes. The construction of time operators associated with the latter processes is the main purpose of this paper.

The plan of the paper is the following. In Section 2, we remaind some basic concepts and links between stationary processes and classical dynamical systems. Then, we present a simple time operator theory for the processes that are stationary in the wide sense. In Section 3, we study the time operator associted with a discrete time gaussian stationary process. This stronger assumption allows to expand significantly the domain of the time operator and to drive the explicit form of its eigenprojectors. The time operator is defined on a dense subspace of the Fock space determined by the gaussian process.

2. TIME OPERATORS OF PROCESSES STATIONARY IN THE WIDE SENSE

We begin this section with a brief reminder of basic concepts and notions from the theory of stochastic processes. The reader will find more details in Doob (1953) and Lamperti (1977).

Let $(\Omega, \mathcal{F}, P)$ be a probability space, I an index set, and $\{X_t\}_{t\in I}$ a real or complex valued stochastic process on $(\Omega, \mathcal{F}, P)$. The index set I is assumed to be either the set of integers or real numbers.

A family $\{\mathcal{F}_t\}_{t\in I}$ of sub-σ-algebras of $\mathcal{F}$ is called to be *adapted* to the process $\{X_t\}_{t\in I}$ (shortly, $\{\mathcal{F}_t\}_{t\in I}$ is $\{X_t\}_{t\in I}$ adapted) if every random variable X_t is $\mathcal{F}_t$ measurable. A family $\{\mathcal{F}_t\}_{t\in I}$ is called a *filtration* of the process $\{X_t\}_{t\in I}$ if:

(1) $\mathcal{F}_s \subset \mathcal{F}_t$, for each $s < t$,
(2) $\{X_t\}$ is $\{\mathcal{F}_t\}$ adapted.

In particular, $\{\mathcal{F}_t\}_{t\in I}$ is called the *natural filtration* of the process $\{X_t\}_{t\in I}$ if each $\mathcal{F}_t$ is defined as the smallest σ-algebra generated by all random variables X_s, for $s \leq t$.

As filtration $\{\mathcal{F}_t\}_{t\in I}$ can be associated with the family of conditional expectations $\{E_t\}$

$$E_t \stackrel{\text{df}}{=} E(\cdot \mid \mathcal{F}_t), \quad t \in I$$

acting on the Hilbert space $L^2(\Omega, \mathcal{F}, P)$.

A stochastic process $\{X_t\}$ is said to be *strictly stationary* or *stationary in a narrow sense* if all finite dimensional distributions of the process $\{X_t\}$ are time invariant, i.e.,

$$P\left\{X_{t_1+s} \in B_1, \ldots, X_{t_n+s} \in B_n\right\} = P\left\{X_{t_1} \in B_1, \ldots, X_t \in B_n\right\}$$

for each $s, t_1, \ldots, t_n \in I$, $n \in \mathbb{N}$, and $B_1, \ldots, B_n$ are Borel subsets of $\mathbb{R}(\mathbb{C})$.

A process $\{X_t\}$ is said to be *stationary in the wide sense* if it is a second-order process ($E|X_t|^2 < \infty$, for each t), the mean value of X_t is constant

$$EX_t = m$$

(in the sequel we shall always assume that $m = 0$) and the covariance function

$$R(s, t) \stackrel{\text{df}}{=} EX_s\bar{X}_t - |m|^2$$

depends only on the difference between s and t

$$R(s+u, t+u) = R(s, t)$$

for each $s, t, u \in \mathbb{R}$.

Note that in the class of L^2-stochastic processes each strictly stationary process is also stationary in the wide sense. Strictly stationary stochastic processes may arise from dynamical systems with measure-preserving transformations. Indeed, let us consider a dynamical system $(\mathcal{X}, \Sigma, \mu; \{S_t\})$, where μ is a normalized measure and each transformation S_t is measure-preserving, i.e., $\mu(S_t^{-1}A) = \mu(A)$, for each $A \in \Sigma$. Then define the probability space $(\Omega, \mathcal{F}, P)$ by putting $\Omega = \mathcal{X}$, $\mathcal{F} = \Sigma$, and $P = \mu$, and the stochastic process

$$X_t(\omega) \stackrel{\text{df}}{=} f(S_t x) \tag{2}$$

where f is some real (or complex) valued measurable function on $\mathcal{X}$. We have

$$\begin{aligned} &P\left\{\omega \in \Omega : X_{t_1+s}(\omega) \in A_1, \ldots, X_{t_n+s}(\omega) \in A_n\right\} \\ &= \mu\left(S_{t_1+s}^{-1} f^{-1}(A_1) \cap \cdots \cap S_{t_n+s}^{-1} f^{-1}(A_n)\right) \\ &= \mu\left(S_{t_1}^{-1} f^{-1}(A_1) \cap \cdots \cap S_{t_n}^{-1} f^{-1}(A_n)\right) \\ &= P\left\{\omega \in \Omega : X_{t_1}(\omega) \in A_1, \ldots, X_{t_n}(\omega) \in A_n\right\} \end{aligned}$$

for any choice $t_1, \ldots, t_n, s \in I$ and Borel sets $A_1, \ldots, A_n$. This shows that $\{X_t\}$ is strictly stationary on I. Choosing as f a square integrable function, we obtain a stochastic process that is stationary both in the strict and in the narrow sense.

Since each measurable function f on the phase space Ω defines a different strictly stationary process, one may say that the dynamical systems with measure-preserving transformations have a more complex structure than strictly stationary stochastic processes. However, it is not true that for an arbitrary strictly stationary process $\{X_t\}$ on the probability space $(\Omega, \mathcal{F}, P)$ there exists a family $\{S_t\}$ of measure-preserving transformation such that (2) holds (see Lamperti, 1977 for the discussion on this subject).

Let us consider first an arbitrary real or complex valued process $\{X_t\}_{t\in I}$ on a probability space $(\Omega, \mathcal{F}, P)$. Let the set I of indices, interpreted here as *time*, be either the real line $\mathbb{R}$ or the set of intergers $\mathbb{Z}$. Let $\{\mathcal{F}_t\}_{t\in I}$ be the natural filtration determined by the process $\{X_t\}_{t\in I}$. We assume in addition that the σ-algebra generated by all X_t coincides with $\mathcal{F}$.

The conditional expectations E_t, regarded as operators on the Hilbert spce L_2, are orthogonal projectors. If the family $\{E_t\}$ is a resolution of identity then we can define the self-adjoint operator T

$$Tf = \int_I t\,dE_t f \tag{3}$$

which is densely defined on L^2.

It is known thast formula (3) defines a time operator in the case when $\{\mathcal{F}_t\}$ is a nested family of an K-system or an exact system. One may therefore ask whether T is defined by (3) is also a time operator of the stochastic process $\{X_t\}$. This question acquires meaning if T can be related through (1) to some semigroup of operators that reflects the dynamics (or the flow of time) consistent witht he filtration determined by $\{X_t\}$. However, an arbitratary stochastic process does not determine any specific dynamics that could be expressed by a semigroup of operators. Only, if the process $\{X_t\}_{t\in I}$ is stationary (in the wide sense), we can proceed as follows.

Let $\mathcal{H}(X)$ denote the closed subspace of L^2 spanned by the linear combinations of $X_t, t \in I$. $\mathcal{H}(X)$ is also a Hilbert space (in general, a proper subspace of L^2). On the space $\mathcal{H}(X)$ we can define the shift operator

$$V_t X_s \stackrel{\text{df}}{=} X_{s+t}$$

extending it by linearity on finite linear combinations of $X_{t_1}, \dots, X_{t_n}$. The operator V_t preserves the L^2-norm

$$\left\| V_t \left(\sum_{j=1}^{n} a_j X_{t_j} \right) \right\|^2 = \left\| \sum_{j=1}^{n} a_j X_{t_j+t} \right\|^2 = \sum_{j,k=1}^{n} a_j \bar{a}_k E X_{t_j+t} \bar{X}_{tk+t}$$

$$= \sum_{j,k=1}^{n} a_j \bar{a}_k E X_{t_j} \bar{X}_{t_k} = \left\| \sum_{j=1}^{n} a_j X_{t_j} \right\|^2$$

In consequence, each V_t extends to an unitary operator on $\mathcal{H}$ (X). In this way, we define the dynamics associated with the stationary process $\{X_t\}$ as the unitary group $\{V_t\}$.

Let $\mathcal{H}_t$ (X) be the closed subspace of L^2 spanned by the linear combinations of X_s, $s \in I$, $s \leq t$. In fact $\mathcal{H}_t(X)$ is a subspace of $L^2(\mathcal{F}_t)$. Note that, for each t, the conditional expectation E_t is the orthogonal projection from $\mathcal{H}$ (X) onto $\mathcal{H}_t$ (X). Recall that the process $\{X_t\}$ ($EX_t = 0$, for each t) is purely nondeterministic if

$$\bigcap_{t\in I} \mathcal{H}_t(X) = \{0\}$$

Proposition 1. *If the process* $\{X_t\}$ *is stationary in the wide sense and purely nondeterministic then the operator* T *associated with* $\{X_t\}$ *through* (3) *is a time operator with respect to* $\{V_t\}$.

In the proof of this proposition, and also in the sequel, we shall use the following simple property of Hilbert spaces:

Lemma 1. *Let $\mathcal{H}_0$ be a closed subspace of the Hilbert space $\mathcal{H}$. If $x_0 \in \mathcal{H}_0$ and $x \in \mathcal{H}$, then*

$$x - x_0 \perp \mathcal{H}_0 \iff x_0 = P_0 x$$

where P_0 is the orthogonal projector on $\mathcal{H}_0$.

Proof of the Proposition 1: By the assumption that $\{X_t\}$ is purely nondeterministic the family $\{E_t\}$ is a resolution of identity in the Hilbert space L^2 as well as in its subspace $\mathcal{H}(X)$. Thus, equality (3) defines a selfadjoint operator on a dense subspace of both $\mathcal{H}(X)$ and L^2. To see that T and $\{V_t\}$ satisfy

$$TV_t = V_t T + tV_t \tag{4}$$

we need the equality

$$V_t E_s = E_{s+t} V_t \tag{5}$$

Let Y be an arbitrary element from $\mathcal{H}(X)$. Then $E_s Y \in \mathcal{H}_s(X)$ and $V_t E_s Y \in \mathcal{H}_{s+t}(X)$. By Lemma 1, $Y - E_s Y \perp \mathcal{H}_s(X)$. Since V_t maps $\mathcal{H}_s$ onto $\mathcal{H}_{s+t}$, we also have $V_t(Y - E_s Y) \perp \mathcal{H}_{s+t}(X)$. Thus

$$V_t Y - V_t E_s Y \perp \mathcal{H}_{s+t}(X)$$

and applying the lemma again we obtain

$$V_t E_s Y = E_{s+t} V_t Y$$

which proves (5). Now, to prove (4) it is enough to use representation (3) of T and apply (5). □

Purely nondeterministic stationary processes correspond to K-systems or exact systems, i.e., to the dynamical systems for which time operators have been already constructed (Misra *et al.*, 1979; Suchanecki and Weron, 1990; Antoniou and Suchanecki, 2000; Antoniou *et al.*, 1999). If $\{X_t\}$ is not purely nondeterministic then it follows from the Wold decomposition theorem that $\{X(t)\}$ can be represented uniquely as

$$X(t) = X_1(t) + X_2(t)$$

where the process $\{X_1(t)\}$ is purely nondeterministic and $\{X_2(t)\}$ is deterministic, i.e., $\mathcal{H}_t(X_2) = \mathcal{H}(X_2)$, for all t. Moreover $\{X_1(t)\}$ and $\{X_2(t)\}$ are L^2-orthogonal. Since the deterministic process is measurable with respect to the σ-algebra $\bigcap_t \mathcal{F}_t$, it is not affected by the transition $\mathcal{F}_s \to \mathcal{F}_t$, for $s < t$. Thus, we can say that there is no flow of time for a deterministic process. Consequently, the time operator must be identity on the space $\mathcal{H}(X_2)$, which is the orthogonal complement of $\mathcal{H}(X_1)$.

Let us now consider a discrete time stationary process, i.e., an L^2-stationary sequence $\{X_n\}_{n\in\mathbb{Z}}$, which is purely nondeterministic. We can characterize the time operator associated with $\{X_n\}$ in terms of its eigenvalues and eigenspaces as follows:

Proposition 2. *Let $X = \{X_n\}_{n\in\mathbb{Z}}$ be a purely nondeterministic stationary sequence. Then there is an orthonormal basis $\{Y_n\}$ in the Hilbert space $\mathcal{H}(X)$ such that the time operator T associated with the process $\{X_n\}$ has the form*

$$T = \sum_{n\in\mathbb{Z}} nP_n \tag{6}$$

where P_n denotes the projection on the space spanned by Y_n.

Proof: The process $\{X_n\}$ can be represented in the form

$$X_n = \sum_{k=-\infty}^{n} a_{n-k}Y_k, \quad n \in \mathbb{Z} \tag{7}$$

where $Y_k \in L^2$, $k \in \mathbb{Z}$, is a sequence of random variables that are mutually orthonormal, $\sum_{k=0}^{\infty} |a_k|^2 < \infty$, and $\mathcal{H}_n(X) = \mathcal{H}_n(\mathrm{Y})$ (see [15]). This implies that each E_n coincides with the orthogonal projection onto $\mathcal{H}_n(Y)$.

Let $P_n \stackrel{\mathrm{df}}{=} E_n - E_{n-1}$. Since E_n is the orthogonal projection on $\mathcal{H}_n(Y)$, P_n is the orthogonal projection on the space $\mathcal{H}_n(Y) \ominus \mathcal{H}_{n-1}(Y)$, which is, by the assumption that $\{X_n\}$ is purely nondeterministic, one dimensional space generated by Y_n. Time operator (3) associated with $\{X_n\}$ can be equivalently represented by (6). □

The process $\{Y_n\}$ from the above proposition is called the *spectral process* or the *white noise*. If the random variables Y_n are independent then we can say that $\{Y_n\}$ is the *innovation process* of $\{X_n\}$ (Antoniou and Suchanecki, 2003) (see also (Hida, 1997; Accardi *et al.*, 2002). Thus, the above theorem connects the time operator with the spectral (or innovation) process.

Using Theorem 1 we may connect time operators with all those processes that can be derived from the spectral (or innovation) processes as their linear functionals. This connection between time operators and innovation processes allows a new interpretation of the operation of *time scaling* in classical dynamical systems.

Time scalings have been widely used in the Misra–Prigogine–Courbage theory of irreversibility, which explains irreversible behaviour of highly unstable, although reversible, dynamical systems. We would like to present now the stochastic interpretation of the time scaling $\Lambda(T)$ of the just constructed time operator T.

First, let us note that for any $Z \in \mathcal{H}(X)$ the value $\langle Z, TZ\rangle$ can be interpreted as the average age of Z. It follows from the spectral resolution

$$T = \sum_{n\in\mathbb{Z}} nP_n \tag{8}$$

that T attributes the age n to the random variable Y_n. Since $\{Y_n\}$ form a complete orthonormal system in the Hilbert space $\mathcal{H}(X)$, each $Z \in \mathcal{H}(X)$ can be represented as

$$Z = \sum_{n\in\mathbb{Z}} b_n Y_n \tag{9}$$

where $\sum_n |b_n|^2 < \infty$. Consequently each element of $\mathcal{H}(X)$ can be decomposed in terms of eigenvectors of T. Moreover, the domain of T consists of all Z of the form (9) for which $\sum_n |b_n|^2 < \infty$. We can then say that each Z from the domain of T has well definite age, which is

$$\langle Z, TZ\rangle = \sum_n n|b_n|^2$$

Now, if

$$\Lambda = \Lambda(T) = \sum_{n\in\mathbb{Z}} \lambda_n P_n \tag{10}$$

is a function of the time operator, then Λ attributes the age λ_n to Y_n.

It is interesting to interpret Λ-operator in terms of filtering theory. From this point of view the sequence $\{Y_k\}$ is a noise and X_n is the response of a linear homogenous physically realizable system at instant n to the sequence of impulses $\{Y_k\}$. According to this interpretation at every time instant n there is the same contribution of noise. Λ-operator changes the magnitude of impulses of the noise. As a result the system becomes inhomogenous with respect to time although still physically realizable.

3. TIME OPERATORS OF STRICTLY STATIONARY PROCESSES—FOCK SPACE

There is a profound difference between the above introduced time operators associated with wide-sense stationary processes and the time operators associated with K-systems. Although T, defined in Section 2, keeps step with the evolution semigroup $\{V_t\}$ the time eigensubspaces are one-dimensional. This is in a sharp contrast with the properties of the time operator that was originally introduced by Misra and Prigogine, which has the eigenfunctions of infinite multiplicity. However, let us notice that formula (3) actually defines T on the whole space $L^2(\mathcal{F}) \stackrel{\text{df}}{=} L^2(\omega, \mathcal{F}, P)$, where $\mathcal{F}$ is the σ-algebra generated by all X_t, $t \in I$. We show below that for some classes of strictly stationary processes there is an extension of the evolution $\{V_t\}$ from $\mathcal{H}(X)$ to the unitary evolution on $L^2(\mathcal{F})$ such that T is a strict analog of the Misra–Prigogine time operator with respect to this extended group. Namely, that the extended unitary group $\{V_t\}$ is such that $L^2(\mathcal{F}_0) \subset V_t L^2(\mathcal{F}_0)$, for $t > 0$, where $\mathcal{F}_0 = \sigma(X_t)_{t<0}$, and has a homogeneous Lebesgue spectrum of infinite multiplicity.

Let us now assume that the stochastic process under consideration is a Gaussian stationary sequence $\{X_n\}_{n\in\mathbb{Z}}$ with $EX_n = 0$, for each $n \in \mathbb{Z}$. In this case $\{X_n\}$ is stationary in both strict and narrow sense and the Hilbert space $\mathcal{H}(X)$ is the space of Gaussian random variables.

The shifts V^m, $V^m X_n = X_{n+m}$, can be extended from $\mathcal{H}(X)$ on all functions $f(X_{n_1}, \ldots, X_{n_k})$ of finite subsequences of $\{X_n\}$ by putting

$$V^m f\big(X_{n_1}, \ldots, X_{n_k}\big) = f\big(X_{n_1+m}, \ldots, X_{n_k+m}\big) \tag{11}$$

It is easy to see that extended V^m is also an isometry on functions of the form $f(X_{n_1}, \ldots, X_{n_k})$. Since such functions generate $L^2(\mathcal{F})$, V^m extends to an isometry on the whole $L^2(\mathcal{F})$.

Denote by $\mathcal{F}_n$ the σ-algebra generated by $X_k, k \le n$, and by E_n the orthogonal projection (conditional expectation) from $L^2(\mathcal{F})$ onto $L^2(\mathcal{F}_n)$. We shall show below that operator T

$$T = \sum_{n\in\mathbb{Z}} n(E_n - E_{n-1}) \tag{12}$$

which a discrete analog of (3), is a time operator associated with the extended unitary group V^m on $L^2(\mathcal{F})\ominus$ (Misra, 1978)—the orthogonal complement of constants in $L^2(\mathcal{F})$.

Let us note that the Hilbert space of square integrable functions $f(X)$ of a Gaussian random variable X can be identified with $L^2(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, \gamma)$, where $\mathcal{B}_{\mathbb{R}}$ is the Borel σ-algebra on $\mathbb{R}$ and γ is the Gaussian measure on $\mathbb{R}$ with the density $(2\pi)^{-1/2}e^{-x^2/2}$. Since $L^2(\mathcal{F})$ actually consists of all functions of the form $f(X_1, X_2, \ldots)$ (see Dynkin, 1961, Lemma 1.5), it can be identified with the infinite product space $L^2(\mathbb{R}^\infty, \mathcal{B}_{\mathbb{R}}^\infty, \gamma_\infty)$, where γ_∞ is the corresponding product Gaussian measure on $\mathbb{R}^\infty$. This identification will allow to express spectral projectors of T in terms of the Wick polynomials.

In the sequel we use the following:

Lemma 2. (Kakutani, 1961; Major, 1981). *If $\{Y_k\}$ is an orthonormal basis in $\mathcal{H}(X)$ then the family of all products*

$$H_{l_1}\big(Y_{j_1}\big) \cdots H_{l_r}\big(Y_{j_r}\big), \tag{13}$$

where $H_n(x)$ denotes the nth Hermite polyomial with the leading coefficient 1, *form a complete orthogonal system in $L^2(\mathcal{F})$.*

Recall that the nth Hermite polynomial with the leading coefficient 1 is

$$H_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2}$$

In particular $H_0 \equiv 1$, $H_1(x) = x$, $H_2(x) = x^2 - 1$, $H_3(x) = x^3 - 3x, \ldots$ Recall also that $H_n(x)$, $n = 0, 1, \ldots,$ form a complete orthogonal basis in $L^2(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, \gamma)$. In fact, each $f \in L^2(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, \gamma)$ can be represented in the form

$$f(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} H_n(x) \tag{14}$$

where $a_n = \langle f, H_n \rangle$.

Let $\{Y_n\}_{n\in\mathbb{Z}}$ be the innovation process corresponding to $\{X_n\}_{n\in\mathbb{Z}}$, i.e., a family of independent $\mathcal{N}(0, 1)$ random variables such that

$$X_n = \sum_{k=-\infty}^{n} a_{n-k} Y_k, \quad n \in \mathbb{Z} \tag{15}$$

Consider the Hilbert space $L^2(\mathcal{F}_n)$. It is easy to see that the σ-algebra $\mathcal{F}_n$ coincides with the smallest σ-algebra generated by Y_k, $k \le n$. Moreover the above lemma implies that products (13) with $j_k \le n$ form a complete orthogonal system in $L^2(\mathcal{F}_n)$. Next, denote by $\mathcal{H}_{\le k}(n)$ the Hilbert space spanned by all polynomials

$$p\big(Y_{j_1}, \ldots, Y_{j_r}\big) = \sum_{l_1,\ldots,l_r} a_{l_1,\ldots,l_r} Y_{j_1}^{l_1} \cdots Y_{j_r}^{l_r} \tag{16}$$

of random variables Y_j with $r = 1, 2, \ldots, j_1, \ldots, j_r \le n$ and $l_1, \ldots, l_r$ non-negative integers with $l_1 + \cdots + l_r \le k$. Let $\mathcal{H}_0(n)$ denote the space of constants and let $\mathcal{H}_k(n)$ be the orthogonal complement of $\mathcal{H}_{\le k-1}(n)$ in $\mathcal{H}_{\le k}(n)$, i.e., $\mathcal{H}_{\le k-1}(n) \oplus \mathcal{H}_k(n) = \mathcal{H}_{\le k}(n)$. Therefore

$$L^2(\mathcal{F}_n) = \bigoplus_{k=0}^{\infty} \mathcal{H}_k(n).$$

Theorem 1. *Suppose that $\{X_n\}_{n\in\mathbb{Z}}$ is a Gaussian stationary sequence on the probability space $(\Omega, \mathcal{F}, P)$ and $\{Y_n\}_{n\in\mathbb{Z}}$ its innovation process* (15). *Assume that $\mathcal{F}$ coincides with the smallest σ-algebra generated by all X_n, $n \in \mathbb{Z}$. Then operator T considered on $L^2(\mathcal{F}) \ominus [1]$:*

$$T = \sum_{n\in\mathbb{Z}} n(E_n - E_{n-1})$$

is a time operator of the group of shifts $\{V^m\}$ defined by (11) *and its projections E_n are of the form*

$$E_n = \sum_{k=1}^{\infty} P_n(k)$$

where $P_n(k)$, $k = 1, \ldots,$ is the orthogonal projecton on the space $\mathcal{H}_k(n)$. In particular, for any finite subsequence $Y_{j_1}, \ldots, Y_{j_r}$, $j_r \le n$, and any homogeneous *polynomial of order k*

$$p\big(Y_{j_1},\ldots,Y_{j_r}\big)=\sum_{l_1,\ldots,l_r} a_{l_1,\ldots,l_r} Y_{j_1}^{l_1},\ldots,Y_{j_r}^{l_r}$$

the orthogonal projecton $P_n(k)$ of $p(Y_{j_1},\ldots,Y_{j_r})$ is of the form

$$P_n(k)\big(p\big(Y_{j_1}\cdots Y_{j_r}\big)\big)=\sum_{l_1,\ldots,l_r} a_{l_1,\ldots l_r} H_{l_1}\big(Y_{j_1}\big),\ldots,H_{l_r}\big(Y_{j_r}\big)$$

where the summation is over all nonnegative integers $l_1+\cdots+l_r=k$.

Proof: It follows from Lemma 2 (see also Kakutani, 1961; Major, 1981) that the orthogonal projection $P_n(k)$ of the polynomial (16) of order k onto $\mathcal{H}_k(n)$ is the Wick polynomial

$$P_n(k)\big(p\big(Y_{j_1},\ldots,Y_{j_r}\big)\big)=\sum_{l_1,\ldots,l_r} a_{l_1,\ldots,l_r} H_{l_1}\big(Y_{j_1}\big),\ldots,H_{l_r}\big(Y_{j_r}\big) \tag{17}$$

Since $H_{l_1}(Y_{j_1}),\ldots,H_{l_r}(Y_{j_r})$ form a complete orthonormal system in $L^2(\mathcal{F}_n)$ and $\cup_{n\in\mathbb{Z}}L^2(\mathcal{F}_n)=L^2(\mathcal{F})$, the explicit form of the spectral projectors of the time operator T is known on a dense subspace of $L^2(\mathcal{F})$.

In order to show that T is the time operator with respect to the extended shift V on $L^2(\mathcal{F})\ominus$ (Misra, 1978) note first that

$$V^m L^2(\mathcal{F}_n)=L^2(\mathcal{F}_{n+m}) \tag{18}$$

Indeed, because $L^2(\mathcal{F}_n)$ is the closure of the space spanned by all $H_{l_1}(Y_{j_1}),\ldots,H_{l_r}$ (Y_{j_r}), with $j_1,\ldots,j_r\le n$, it follows from (11) that

$$V^m\big(H_{l_1}\big(Y_{j_1}\big),\ldots,H_{l_r}\big(Y_{j_r}\big)\big)=H_{l_1}\big(Y_{j_1+m}\big),\ldots,H_{l_r}\big(Y_{j_r+m}\big)$$

Since the right-hand sides of the latter equality spans $L^2(\mathcal{F}_{n+m})$, this concludes the proof of (18).

Finally, arguing in the same way as in the proof of Proposition 1 we show that T is a time operator of the group of shifts $\{V_m\}$. This concludes the proof of the theorem. □

Remark 1. The space $L^2(\mathcal{F})$ can be also decomposed as a direct sum

$$L^2(\mathcal{F})=\bigoplus_{k=0}^{\infty}\mathcal{H}_k$$

in the same manner as the spaces $L^2(\mathcal{F}_m)$. It follows from Proposition 2 and (14) that each $f\in\mathcal{H}_k$ is of the form .

$$f=\sum a_{l_1,\ldots,l_r} H_{l_1}\big(Y_{j_1}\big),\ldots,H_{l_r}\big(Y_{j_r}\big)$$

where $r = 1, 2, \ldots, j_1, \ldots, j_r \in \mathbb{Z}$, and $l_1, \ldots, l_r$ are positive integers with $l_1 + \cdots + l_r = k$. The norm of f in $L^2(\mathcal{F})$ is

$$\|f\|^2 = \sum \left|a_{l_1,\ldots,l_r}\right|^2 \frac{1}{l_1!..l_r!}$$

Note also that each space $\mathcal{H}_k$ is invariant with respect to the group $\{V^m\}$ of extended shifts

$$\|V^m f\|^2 = \sum a_{l_1,\ldots,l_r} H_{l_1}\left(Y_{j_1+m}\right), \ldots, H_{l_r}\left(Y_{j_r+m}\right)$$

Remark 2. In the case of a continuous time stationary process the representation of the corresponding time operator in terms of its eigenvalues and eigenprojectors is more eleborated. In order to obtain a direct analog of (7) the spectral (or innovative process) $\{Y_n\}$ has to be replaced by a generalized stochastic process understand as a family of linear functionals on a topological vector space (Hida and Ikeda, 1967; Hida, 1970). An analog of Theorem 1 can be obtained by considering representations of eigenprojectors in terms of a (generalized) innovation process. This approach requires, however, the introduction of new tools and will be presented in a separate publication (Antoniou *et al.*, in press).

REFERENCES

Accardi, L., Hida, T., and Si, S. (2002). Innovation approach to some stochastic processes. *Volterra Center Notes*, N. 537.

Antoniou, I. and Gustafson, K. (1999). Wavelets and stochastic processes. *Math. Comp. Simul.* **49**, 81–104.

Antoniou, I. and Gustafson, K. (2000). The time operator of wavelets. *Chaos, Solitons and Fractals* **11**, 443–452.

Antoniou, I. and Suchanecki, Z. (2000). Non-uniform time operator, chaos and wavelets on the interval. *Chaos Solitons and Fractals* **11**, 423–435.

Antoniou, I. and Suchanecki, Z. (2003a). Time operators associated to dilations of Markov processes. In *Progress in Nonlinear Differential Equations and their Applications*, Vol. 55, Birkhäuser, pp. 13–23.

Antoniou, I. and Suchanecki, Z. (2003b). Internal time and innovation. In *The Nature and Time: Geometry, Physics and Perception*, Eds. R. Buccheri, M. Saniga, and W. M. Stuckey, NATO Science Series II, Vol. 95, Kluwer Academic Publishers, Dordrecht, pp. 197–208.

Antoniou, I., Sadovnichii, V. A., and Shkarin, S. A. (1999). Time operators and shift representation of dynamical systems. *Physica A*, **269**, 299–313.

Antoniou, I., Prigogine, I., Sadovnichii, V. A., and Shkarin, S. A. (2000). Time operator for diffusion. *Chaos, Solitons and Fractals* **11**, 465–477.

Antoniou, I., Hida, T., Si, S., and Suchanecki, Z. (in press). Innovations and time operators of stationary processes.

Doob, J. L. (1953). *Stochastic Processes*, Wiley, New York.

Dynkin, E. B. (1961). *Die Grundlagen der Theorie der Markoffschen Prozesse,* Springer Verlag, Berlin-Göttingen-Heidelberg.

Hida, T. (1970). *Stationary Stochastic Processes*, Princeton University Press.

Hida, T. (1997). Random fields and quantum dynamics. *Foundations of Physics* **27**, 1511–1518.

Hida, T. and Ikeda, N. (1967). Analysis on Hilbert space with reproducing kernel arising from multiple Wiener integral. In *Proceedings of the Fifth Berkeley Symposium on Mathematical Statistics and Probability*, Vol. 2, pp. 117–143.

Kakutani, S. (1961). Spectral analysis of stationary Gaussian processes. *Proc. 4th Berkeley Symp. Math. Stat. Prob.* **2**, 239–247.

Lamperti, J. (1977). *Stochastic Processes. A Survey of the Mathematical Theory,* Springer-Verlag, New York Inc.

Major, P. (1981). *Multiple Wiener-Itô Integrals*, Lecture Notes in Mathematics 849, Springer-Verlag.

Misra, B. (1978). Nonequilibrium entropy, Lyapunov variables, and ergodic properties of classical systems. *Prac. Natl. Acad.* USA **75**, 1627–1631.

Misra, B., Prigogine, I., and Courbage, M. (1979). From deterministic dynamics to probabilistic descriptions. *Proc. Natl. Acad. Sci. USA* **76**, 3607–3611.

Prigogine, I. (1980). *From Being to Becoming,* Freeman, New York.

Suchanecki, Z. (1992). On lambda and internal time operators. *Physica A* **187**, 249–266.

Suchanecki, Z. and Weron, A. (1990). Applications of an operator stochastic integral in Prigogine's theory of irreversible dynamical systems. *Expo. Math.* **8**, 67–69.

Suchanecki, Z. and Antoniou, I. (2003). Time operators, innovations and approximations. *Chaos Solitons and Fractals* **17**, 337–342.